A uniform estimate for rough paths

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Abstract

We prove an extension to the classical continuity theorem in rough paths (Theorem 2.2.2 in [10]). We show that two p-rough paths are close in all levels of iterated integrals provided the first $\lfloor p \rfloor$ terms are close in a uniform sense. Applications include estimation of the difference of the signatures of two uniformly close paths ([12]) and convergence rates of Gaussian rough paths ([13]).

1 Introduction

1.1 Motivation

The classical continuity theorem in rough paths (Theorem 2.2.2 in [10]) states that if X and Y are two p-rough paths whose p-variation are both controlled by ω and such that

$$\left\| X_{s,t}^{k} - Y_{s,t}^{k} \right\| \leq \epsilon \frac{\omega(s,t)^{\frac{n}{p}}}{\beta(\frac{n}{p})!}, \qquad k = 1, \cdots, \lfloor p \rfloor, \tag{1}$$

then (1) holds for all $n \geq 1$. The proof is by an induction argument, which depends on the value of the exponent on the control, namely $\frac{n}{p}$. Although it is powerful in many places, there are certain problems for which we need a more convenient (and weaker) assumption. More precisely, we assume

$$\left\| X_{s,t}^{k} - Y_{s,t}^{k} \right\| < \epsilon \frac{\omega(s,t)^{\frac{k-\delta}{p}}}{\beta(\frac{k}{p})!}, \qquad k = 1, \cdots, \lfloor p \rfloor. \tag{2}$$

where $\delta \in [0,1]$. We wish to study whether similar estimates hold for $n \geq \lfloor p \rfloor$. It is easy to see that the classical assumption (1) corresponds to $\delta = 0$.

Such estimates are useful in a number of problems. For example, consider the following two linear differential equations

$$dx_t = Ax_t d\gamma_t, \qquad x_0 = a, \tag{3}$$

and

$$dy_t = Ax_t d\tilde{\gamma}_t, \qquad y_0 = a, \tag{4}$$

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where $\gamma, \tilde{\gamma} : [0,1] \to \mathbb{R}^d$ are two paths of bounded variations whose lengths are both controlled by ω. Suppose further that

$$\sup_{t \in [0,1]} |\gamma_t - \tilde{\gamma}_t| < \epsilon, \tag{5}$$

and one wishes to estimate the difference of the solution flow $|x_t - y_t|$. This question involves estimating the differences between all higher degrees of iterated integrals of γ and $\tilde{\gamma}$, which are called signatures (we will give a precise definition in the next section). If we let X and Y to be the signatures of γ and $\tilde{\gamma}$, then assumption (5) can be written as

$$\left\| X_{s,t}^1 - Y_{s,t}^1 \right\| \leq 2\epsilon = 2\epsilon\omega(s,t)^0.$$

We see that it falls in the assumption (2) with $p = 1$ and $\delta = 1$. We will come back to this question at the end of this paper.

Our estimates also apply to obtaining the convergence rates of Gaussian rough paths. For details, we refer to the recent work [13].

Notation. In what follows, p will always be a number that is at least 1. We use $\lfloor p \rfloor$ to denote the largest integer that does not exceed p, and let $\{p\} = p - \lfloor p \rfloor$ to be the fracal part of p.

1.2 Main results

Before stating our main result, let us explain briefly why the induction argument for the classical continuity theorem does not work directly here. As mentioned earlier, the induction argument depends on the exponent $\frac{n}{p}$. More precisely, the exponent for the level $n + 1 = \lfloor p \rfloor + 1$ is expected to be

$$\frac{\lfloor p \rfloor + 1}{p} > 1. \tag{6}$$

This ensures that when one repeats Young's trick of dropping points, the total sum will converge. However, this condition is not satisfied in our problem (2) unless $\delta < 1 - \{p\}$.

To this point, one may wonder whether one can immediately get the estimate by raising the control to an appropriate power so that the new control satisfies assumption (1). Unfortunately this does not work, for there is no fixed power that one can do it in a homogeneous way for all $k \leq \lfloor p \rfloor$. Furthermore, the new control will in general fail to be superadditive.

The idea is that we will compute one more term by hand, namely the level $\lfloor p \rfloor + 1$. After obtaining the estimate for this term, the exponent on the control will satisfy condition (6), and we can use the usual induction argument for higher levels.

Below is the main theorem of this paper.

Theorem 1.1. *Let $p > 1$, and $\{p\} = p - \lfloor p \rfloor$ be the fractional part of p. Let X, Y be two multiplicative functionals with finite p-variation which are both controlled by*

ω with the same constant β. That is to say: $\forall s,t \in [0,1]$ and $\forall n = 1, \cdots, \lfloor p \rfloor$, we have

$$\left\|X^n_{s,t}\right\| \le \frac{\omega(s,t)^{\frac{n}{p}}}{\beta(\frac{n}{p})!}, \qquad \left\|Y^n_{s,t}\right\| \le \frac{\omega(s,t)^{\frac{n}{p}}}{\beta(\frac{n}{p})!}. \tag{7}$$

Suppose further that there exists an $\epsilon < 1$ such that $\forall k = 1, \cdots, \lfloor p \rfloor$, we have

$$\left\|X^k_{s,t} - Y^k_{s,t}\right\| < \epsilon \cdot \frac{\omega(s,t)^{\frac{k-\delta}{p}}}{\beta(\frac{k}{p})!},$$

where $\delta \in [0,1]$. Then, the followings hold for all $(s,t) \in \Delta$:

1. *If $\delta \in [0, 1-\{p\})$, and β satisfies*

$$\beta > p/[1 - (\frac{1}{2})^{\frac{1-\{p\}-\delta}{p}}], \tag{8}$$

then for all $k \ge 1$, we have

$$\left\|X^k_{s,t} - Y^k_{s,t}\right\| \le \epsilon \cdot \frac{\omega(s,t)^{\frac{k-\delta}{p}}}{\beta(\frac{k}{p})!}. \tag{9}$$

2. *If $\delta = 1 - \{p\}$, and β satisfies*

$$\beta > \frac{4p \cdot 2^{\frac{1-\{p\}}{p}}}{1 - (\frac{1}{2})^{\frac{1-\{p\}}{p}}}, \tag{10}$$

then for all $k \ge \lfloor p \rfloor + 1$, we have

$$\left\|X^k_{s,t} - Y^k_{s,t}\right\| < \epsilon \left(1 + \frac{p}{1-\{p\}} + \log_2 \frac{\omega(0,1)}{\epsilon^{(1-\{p\})/p}}\right) \frac{\omega(s,t)^{\frac{k-1+\{p\}}{p}}}{\beta(\frac{k}{p})!}. \tag{11}$$

3. *If p is non-integer, $\delta \in (1-\{p\}, 1]$, and β satisfies*

$$\beta > 2p\left[\frac{2^{(2p+\delta)/p}}{1 - (\frac{1}{2})^{(\delta-1+\{p\})/p}} + \frac{1}{1 - (\frac{1}{2})^{(1-\{p\})/p}}\right], \tag{12}$$

then for all $k \ge \lfloor p \rfloor + 1$, we have

$$\left\|X^k_{s,t} - Y^k_{s,t}\right\| < \epsilon^{\frac{1-\{p\}}{\delta}} \cdot \frac{\omega(s,t)^{\frac{k-1+\delta}{p}}}{\beta(\frac{k}{p})!}. \tag{13}$$

We see that the assumption becomes weaker as δ increases from 0 to 1. If $\delta < 1 - \{p\}$, then there is no essential differnce with the classical theorem, and the rate ϵ is maintained for all higher levels. If $\delta > 1 - \{p\}$. On the contrary, if $\delta > 1 - \{p\}$, then we have a lost in the power of ϵ in higher levels. In the borderline case where $\delta = 1 - \{p\}$, we get a logorithmic correction.

Remark 1.2. *The case $p = 1$ and $\delta = 1$ was studied in [12], and it was shown that the logarithmic correction can be removed in this situation. But the method there only works for $p = 1$, and does not generalize to other p's.*

1.3 Structure of the paper

This paper is organized as follows. In section 2, we introduce the concepts and notations from rough path theory that are necessary for our current problem. Sections 3 and 4 are devoted to the proof of the main theorem. In section 5, we come back to the example mentioned in the motivation, and explain how our estimates apply to this problem.

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2 Elements from rough path theory

In this section, we introduce some concepts and notations from rough path theory that are needed for the current problem.

Fix the time interval $[0,1]$. For any $0 \leq s < t \leq 1$, write $\Delta_{s,t} = \{(u_1, u_2) | s < u_1 < u_2 < t\}$. In case $(s,t) = (0,1)$, we will simply write $\Delta = \Delta_{0,1}$. For every integer N, write

$$T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes N}.$$

If $\gamma : [0,1] \to \mathbb{R}^d$ is a path of bounded variation, then the signature of γ is defined by

$$X_{s,t}(\gamma) = 1 + X_{s,t}^1 + \cdots + X_{s,t}^n + \cdots ,$$

where

$$X_{s,t}^n = \int_{s<u_1<\cdots<u_n<t} d\gamma_{u_1} \otimes \cdots d\gamma_{u_n} \in T^n(\mathbb{R}^d).$$

It is well known that X is a multiplicative functional, in the sense that for any $s < u < t$, we have

$$X_{s,u} \otimes X_{u,t} = X_{s,t}.$$

If we restrict to the truncated tensor $(1, X_{s,t}^1, \cdots, X_{s,t}^n)$, then it is a multiplicative functional in T^n.

Definition 2.1. *A function $\omega : \Delta \to \mathbb{R}^+$ is a control if it is continuous in both entries, vanishes on the diagonal and superadditive in the sense that for all $s < u < t$, we have*

$$\omega(s,u) + \omega(u,t) \leq \omega(s,t).$$

Definition 2.2. *Let $X : \Delta \to T^n$ be a multiplicative functional. We say X has finite p-variation controlled by ω with a constant β if for all $0 < s < t < 1$ and all $k = 1, \cdots, n$, we have*

$$\left\| X_{s,t}^k \right\| \leq \frac{\omega(s,t)^{\frac{k}{p}}}{\beta(\frac{k}{p})!}.$$

A p-rough path is a multiplicative functional in $T^{\lfloor p \rfloor}$ with finite p-variation controlled by some ω with a constant β. Given a multiplicative functional in T^n, it is natural to ask whether it has a multiplicative extension to T^m for $m > n$. The following (Theorem 2.2.1. in [10]) answers it in affirmative provided it has finite p-variation controlled by some ω.

Theorem 2.3. *Suppose $X : \Delta \to T^n$ is a multiplicative functional in T^n with finite p-variation controlled by ω with constant β, where $n \geq \lfloor p \rfloor$ and*

$$\beta \geq \frac{p}{1 - (\frac{1}{2})^{\frac{\lfloor p \rfloor + 1}{p} - 1}},$$

then for any $m > n$, X has a unique multiplicative extension to T^m with finite p-variation controlled by ω with the same β.

3 Some preliminary lemmas

We first introduce some notations. Let $X_{s,t} = (1, X_{s,t}^1, \cdots, X_{s,t}^n) \in T^{(n)}$ be a multiplicative functional with finite p-variation controlled by ω, where $n \geq \lfloor p \rfloor$. Define

$$\hat{X}_{s,t} = (1, X_{s,t}^1, \cdots, X_{s,t}^n, 0) \in T^{(n+1)}.$$

For any partition $\mathcal{P} = \{s = u_0 < u_1 < \cdots < u_{N-1} < u_N = t\}$, define

$$\hat{X}_{s,t}^{\mathcal{P}} := \hat{X}_{s,u_1} \otimes \cdots \otimes \hat{X}_{u_{N-1},t} \in T^{(n+1)}.$$

The following lemma gives a construction of the unique multiplicative extension of X to higher degrees. It was proved in Theorem 2.2.1. in [10].

Lemma 3.1. *Let $X = (1, X_{s,t}^1, \cdots, X_{s,t}^n)$ be a multiplicative functional in $T^{(n)}$. Let $\mathcal{P} = \{s = u_0 < \cdots < u_N = t\}$ be any parition of (s,t), and $\mathcal{P}^j$ be the partition of (s,t) obtained by removing u_j from $\mathcal{P}$. Then,*

$$\hat{X}_{s,t}^{\mathcal{P}} - \hat{X}_{s,t}^{\mathcal{P}^j} = (0, \cdots, 0, \sum_{k=1}^{n} X_{u_{j-1},u_j}^k \otimes X_{u_j,u_{j+1}}^{n+1-k}) \in T^{(n+1)}.$$

Suppose further that X has finite p-variation controlled by ω, and $n \geq \lfloor p \rfloor$. Then, the limit

$$\lim_{\|\mathcal{P}\| \to 0} \hat{X}_{s,t}^{\mathcal{P}} \in T^{(n+1)}$$

exists. Furthermore, it is the unique multiplicative extension of X to $T^{(n+1)}$, and is also controlled by ω.

Definition 3.2. *A partition $\mathcal{P}$ is a K-dyadic partition of (s,t) with respect to the control ω if*

$$\mathcal{P} = \{s = u_0 < u_1 < \cdots < u_{2^K - 1} < u_{2^K} = t\},$$

and for all $j = 1, 3, 5, \cdots, 2^K - 1$, we have

$$\omega(u_{j-1}, u_j) = \omega(u_j, u_{j+1}).$$

Definition 3.3. *A partition $\mathcal{P}$ is a total K-dyadic partition of $[s,t]$ with respect to the control ω if for each $m = 0, 1, 2, \cdots, K$, the subpartition*

$$\mathcal{P}_{K-m} = \{s = u_0 < u_{2^m} < u_{2^m.2} < \cdots < u_{2^m.2^{K-m}} = t\}$$

is a $(K-m)$-dyadic partition of $[s,t]$ with respect to ω.

Lemma 3.4. *Let $\omega : \Delta \to \mathbb{R}^+$ be a control. For any interval $[s,t]$, for each integer K, there exists a unique total K-dyadic partition $\mathcal{P}_K$ of $[s,t]$ with respect to ω. Furthermore, $\mathcal{P}_{K+1}$ can be obtained from $\mathcal{P}_K$ by inserting one single point between every two consecutive points in $\mathcal{P}_K$ in a unique manner.*

Proof. Since ω is continuous and strictly monotone in both variables, there exists a unique point $u \in (s,t)$ such that

$$\omega(s,u) = \omega(u,t).$$

Thus, $\mathcal{P}_1 = \{s < u < t\}$ is the unique total 1-dyadic partition. Suppose

$$\mathcal{P}_K = \{s = u_0 < u_1 < \cdots < u_{2^K-1} < u_{2^K} = t\}$$

is the unique total K-dyadic partition of $[s,t]$ with respect to ω. Then, for every $u_j < u_{j+1} \in \mathcal{P}_K$, there exists a unique point $v_{j+1} \in (u_j, u_{j+1})$ such that

$$\omega(u_j, v_{j+1}) = \omega(v_{j+1}, u_{j+1}).$$

Thus,

$$\mathcal{P}_{K+1} = \{s = u_0 < v_1 < u_1 < \cdots < u_{2^K-1} < v_{2^K} < u_{2^K} = t\}$$

is the desired unique total dyadic-K partition of (s,t) with respect to ω. □

The next lemma is crucial for our estimates. It was first proved in [10] with a constant $\frac{1}{p^2}$ on the left hand side. Recently, Hara and Hino improved it to $\frac{1}{p}$ in [9].

Lemma 3.5. *(Neo-classical inequality) Let $p \geq 1$, and define $x! := \Gamma(x+1)$. Then for any $x, y \in \mathbb{R}$, we have*

$$\frac{1}{p}\sum_{k=0}^{n} \frac{x^{\frac{k}{p}}}{(\frac{k}{p})!} \cdot \frac{y^{\frac{n-k}{p}}}{(\frac{n-k}{p})!} \leq \frac{(x+y)^{\frac{n}{p}}}{(\frac{n}{p})!}.$$

Lemma 3.6. *Let X, Y be two multiplicative functionals with finite p-variation both controlled by ω with the same constant β. Suppose further that there is an $\epsilon < 1$ such that*

$$\left\| X^k_{s,t} - Y^k_{s,t} \right\| < \epsilon \cdot \frac{\omega(s,t)^{\frac{k-\delta}{p}}}{\beta(\frac{k}{p})!}$$

for each $k = 1, 2, \cdots, n$, where $n \geq \lfloor p \rfloor$, and $\delta \in [0,1]$. Then, we have

$$\left\| (\hat{X}^{\mathcal{P}_{K+1}}_{s,t} - \hat{Y}^{\mathcal{P}_{K+1}}_{s,t})^{n+1} \right\| - \left\| (\hat{X}^{\mathcal{P}_K}_{s,t} - \hat{Y}^{\mathcal{P}_K}_{s,t})^{n+1} \right\|$$
$$\leq \min\left\{ \frac{\epsilon p}{\beta^2(\frac{n+1}{p})!} \cdot 2^{\frac{2p+\delta}{p}} \left(\frac{1}{2^K}\right)^{\frac{n+1-p-\delta}{p}} \omega(s,t)^{\frac{n+1-\delta}{p}}, \quad \left(\frac{1}{2^K}\right)^{\frac{n+1}{p}-1} \frac{2p\omega(s,t)^{\frac{n+1}{p}}}{\beta^2(\frac{n+1}{p})!} \right\}.$$

for all $K = 0, 1, 2, \cdots$.

Proof. For any partition $\mathcal{P} = \{s = u_0 < \cdots < u_j < \cdots < u_N = t\}$, let $\mathcal{P}^j$ be the partition with the point u_j removed from $\mathcal{P}$. By Lemma 3.1, we have

$$\begin{aligned}&(\hat{Y}^{\mathcal{P}}_{s,t} - \hat{Y}^{\mathcal{P}^j}_{s,t})^{n+1} - (\hat{X}^{\mathcal{P}}_{s,t} - \hat{X}^{\mathcal{P}^j}_{s,t})^{n+1}\\ &= \sum_{k=1}^{n}(R^k_{u_{j-1},u_j} \otimes X^{n+1-k}_{u_j,u_{j+1}} + X^k_{u_{j-1},u_j} \otimes R^{n+1-k}_{u_j,u_{j+1}} + R^k_{u_{j-1},u_j} \otimes R^{n+1-k}_{u_j,u_{j+1}}),\end{aligned}$$

where

$$R_{s,t} = Y_{s,t} - X_{s,t}.$$

Thus, we have

$$\begin{aligned}\left\|(\hat{X}^{\mathcal{P}}_{s,t} - \hat{Y}^{\mathcal{P}}_{s,t})^{n+1}\right\| &= \left\|(\hat{X}^{\mathcal{P}^j}_{s,t} - \hat{Y}^{\mathcal{P}^j}_{s,t})^{n+1} + (\hat{X}^{\mathcal{P}}_{s,t} - \hat{X}^{\mathcal{P}^j}_{s,t})^{n+1} - (\hat{Y}^{\mathcal{P}}_{s,t} - \hat{Y}^{\mathcal{P}^j}_{s,t})^{n+1}\right\|\\ &\le \left\|(\hat{X}^{\mathcal{P}^j}_{s,t} - \hat{Y}^{\mathcal{P}^j}_{s,t})^{n+1}\right\| + \sum_{k=1}^{n}\Bigg(\left\|R^k_{u_{j-1},u_j} \otimes X^{n+1-k}_{u_j,u_{j+1}}\right\|\\ &+ \left\|X^k_{u_{j-1},u_j} \otimes R^{n+1-k}_{u_j,u_{j+1}}\right\| + \left\|R^k_{u_{j-1},u_j} \otimes R^{n+1-k}_{u_j,u_{j+1}}\right\|\Bigg).\end{aligned}$$

If $\mathcal{P} = \mathcal{P}_{K+1}$ and $u_j \in \mathcal{P}_{K+1} - \mathcal{P}_K$, then

$$\omega(u_{j-1}, u_j) = \omega(u_j, u_{j+1}) \le \frac{1}{2^{K+1}}\omega(s,t).$$

Thus, for the first term in the above bracket, we have

$$\begin{aligned}\sum_{k=1}^{n}\left\|R^k_{u_{j-1},u_j} \otimes X^{n+1-k}_{u_j,u_{j+1}}\right\| &\le \sum_{k=1}^{n}\epsilon \cdot \frac{\omega(u_{j-1},u_j)^{\frac{k-\delta}{p}}}{\beta(\frac{k}{p})!} \cdot \frac{\omega(u_j,u_{j+1})^{\frac{n+1-k}{p}}}{\beta(\frac{n+1-k}{p})!}\\ &\le \frac{\epsilon}{\beta^2}\left[\frac{1}{2^{K+1}}\omega(s,t)\right]^{\frac{n+1-\delta}{p}}\sum_{k=1}^{n+1}\frac{1}{(\frac{k}{p})!(\frac{n+1-k}{p})!}\\ &\le \frac{\epsilon p}{\beta^2(\frac{n+1}{p})!} \cdot 2^{\frac{\delta}{p}}\left(\frac{1}{2^K}\right)^{\frac{n+1-\delta}{p}}\omega(s,t)^{\frac{n+1-\delta}{p}}.\end{aligned}$$

The same bound holds for the second term. For the third term, note that $\|R\| \le \|X\| + \|Y\|$, thus twice of the previous bound works. By combining bounds for the three terms above, we have

$$\left\|(\hat{X}^{\mathcal{P}_{K+1}}_{s,t} - \hat{Y}^{\mathcal{P}_{K+1}}_{s,t})^{n+1}\right\| \le \left\|(\hat{X}^{\mathcal{P}^j_{K+1}}_{s,t} - \hat{Y}^{\mathcal{P}^j_{K+1}}_{s,t})^{n+1}\right\| + \frac{\epsilon p \cdot 2^{\frac{2p+\delta}{p}}}{\beta^2(\frac{n+1}{p})!}\left[\frac{1}{2^K}\omega(s,t)\right]^{\frac{n+1-\delta}{p}},$$

where u_j is any point in $\mathcal{P}_{K+1} - \mathcal{P}_K$. By successively dropping the 2^K points in $\mathcal{P}_{K+1} - \mathcal{P}_K$, we have

$$\left\|(\hat{X}^{\mathcal{P}_{K+1}}_{s,t} - \hat{Y}^{\mathcal{P}_{K+1}}_{s,t})^{n+1}\right\| \le \left\|(\hat{X}^{\mathcal{P}_K}_{s,t} - \hat{Y}^{\mathcal{P}_K}_{s,t})^{n+1}\right\| + \frac{\epsilon p \cdot 2^{\frac{2p+\delta}{p}}}{\beta^2(\frac{n+1}{p})!}\left(\frac{1}{2^K}\right)^{\frac{n+1-p-\delta}{p}}\omega(s,t)^{\frac{n+1-\delta}{p}}. \tag{14}$$

On the other hand, we have the bound

$$\begin{aligned}\left\|(\hat{X}^{\mathcal{P}_{K+1}}_{s,t} - \hat{X}^{\mathcal{P}^j_{K+1}}_{s,t})^{n+1}\right\| &\le \sum_{k=1}^{n}\left\|X^k_{u_{j-1},u_j} \otimes X^{n+1-k}_{u_j,u_{j+1}}\right\|\\ &\le \frac{p}{\beta^2(\frac{n+1}{p})!}\left[\frac{1}{2^K}\omega(s,t)\right]^{\frac{n+1}{p}},\end{aligned}$$

and the same bound holds for Y. Thus, we have

$$\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}}-\hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\|\leq\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}^{j}}-\hat{Y}_{s,t}^{\mathcal{P}_{K+1}^{j}})^{n+1}\right\|+\frac{2p}{\beta^{2}(\frac{n+1}{p})!}\left[\frac{1}{2^{K}}\omega(s,t)\right]^{\frac{n+1}{p}}.$$

Again, by successively dropping the 2^K points in $\mathcal{P}_{K+1}-\mathcal{P}_K$, we get

$$\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}}-\hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\|\leq\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K}}-\hat{Y}_{s,t}^{\mathcal{P}_{K}})^{n+1}\right\|+\left(\frac{1}{2^{K}}\right)^{\frac{n+1}{p}-1}\frac{2p\omega(s,t)^{\frac{n+1}{p}}}{\beta^{2}(\frac{n+1}{p})!}. \quad (15)$$

Combining (14) and (15), we conclude the lemma.

□

4 Proof of Theorem 1.1

Proof. We fix $s<t$. If $\epsilon\geq 2\omega(s,t)^{\frac{\delta}{p}}$, then (7) automatically implies the theorem. So we may assume without loss of generality that $\epsilon<2\omega(s,t)^{\frac{\delta}{p}}$. In what follows, we let N to be the unique integer such that

$$\left[\frac{1}{2^{N}}\omega(s,t)\right]^{\frac{\delta}{p}}\leq\frac{\epsilon}{2}<\left[\frac{1}{2^{N-1}}\omega(s,t)\right]^{\frac{\delta}{p}}. \quad (16)$$

Case 1. $\delta\in[0,1-\{p\})$.

In this situation, we prove (9) directly by induction. Suppose the theorem holds for $k=1,2,\cdots,n$, where $n\geq\lfloor p\rfloor$. Then, by Lemma 3.6, we have

$$\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}}-\hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\|\leq\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K}}-\hat{Y}_{s,t}^{\mathcal{P}_{K}})^{n+1}\right\|+\frac{\epsilon p\cdot 2^{\frac{2p+\delta}{p}}}{\beta^{2}(\frac{n+1}{p})!}\left(\frac{1}{2^{K}}\right)^{\frac{n+1-p-\delta}{p}}\omega(s,t)^{\frac{n+1-\delta}{p}},$$

and by construction of multiplicative functionals, we have

$$\begin{aligned}\|X_{s,t}^{n+1}-Y_{s,t}^{n+1}\| &= \sum_{K=0}^{+\infty}\left(\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}}-\hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\|-\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K}}-\hat{Y}_{s,t}^{\mathcal{P}_{K}})^{n+1}\right\|\right)\\ &\leq \epsilon\cdot\frac{2^{\frac{2p+\delta}{p}}p}{\beta^{2}(\frac{n+1}{p})!}\omega(s,t)\sum_{K=0}^{+\infty}\left(\frac{1}{2^{K}}\right)^{\frac{n+1-p-\delta}{p}}\\ &\leq \epsilon\cdot\frac{\omega(s,t)^{\frac{n+1-\delta}{p}}}{\beta(\frac{n+1}{p})!},\end{aligned}$$

where the last inequality holds because $n+1-p-\delta>0$ and β satisfies (8).

Case 2. $\delta=1-\{p\}$.

We first prove (11) for level $\lfloor p \rfloor+1$, and after that we can do induction. Applying Lemma 3.6 to $n+1 = \lfloor p \rfloor + 1$, we get

$$\begin{aligned}\left\|(\hat{X}_{s,t}^{\mathcal{P}_N} - \hat{Y}_{s,t}^{\mathcal{P}_N})^{n+1}\right\| &= \sum_{K=0}^{N-1} \left(\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\| - \left\|(\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n+1}\right\| \right) \\ &\leq \frac{\epsilon p N}{\beta^2(\frac{n+1}{p})!} \cdot 2^{\frac{2p+1-\{p\}}{p}} \cdot \omega(s,t).\end{aligned}$$

Note that $2^{N-1} < 2^{\frac{p}{1-\{p\}}}\omega(s,t)/\epsilon^{\frac{p}{1-\{p\}}}$, we have

$$\left\|(\hat{X}_{s,t}^{\mathcal{P}_N} - \hat{Y}_{s,t}^{\mathcal{P}_N})^{n+1}\right\| \leq p \cdot 2^{\frac{2p+1-\{p\}}{p}} \cdot \epsilon \left(1 + \frac{p}{1-\{p\}} + \log_2 \frac{\omega(s,t)}{\epsilon^{(1-\{p\})/p}}\right) \frac{\omega(s,t)}{\beta^2(\frac{n+1}{p})!}. \tag{17}$$

On the other hand, Lemma 3.6 also implies that

$$\sum_{K=N}^{+\infty} \left(\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\| - \left\|(\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n}\right\| \right) \leq \frac{2p\omega(s,t)^{\frac{n+1}{p}}}{\beta^2(\frac{n+1}{p})!} \sum_{K=N}^{+\infty} (\frac{1}{2^K})^{\frac{1-\{p\}}{p}}.$$

Using $\frac{1}{2^N} \leq (\frac{\epsilon}{2})^{\frac{p}{1-\{p\}}}/\omega(s,t)$, we have

$$\sum_{K=N}^{+\infty} \left(\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n}\right\| - \left\|(\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n}\right\| \right) \leq \frac{2p\epsilon}{1-(\frac{1}{2})^{\frac{1-\{p\}}{p}}} \cdot \frac{\omega(s,t)}{\beta^2(\frac{n+1}{p})!}. \tag{18}$$

Combining (10),(17) and (18), we get

$$\begin{aligned}\|X_{s,t}^n - Y_{s,t}^n\| &= \left\|(\hat{X}_{s,t}^{\mathcal{P}_N} - \hat{Y}_{s,t}^{\mathcal{P}_N})^{n}\right\| + \sum_{K=N}^{+\infty} \left(\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n}\right\| - \left\|(\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n}\right\| \right) \\ &< \epsilon \left(1 + \frac{p}{1-\{p\}} + \log_2 \frac{\omega(s,t)}{\epsilon^{(1-\{p\})/p}}\right) \frac{\omega(s,t)}{\beta(\frac{n+1}{p})!}.\end{aligned}$$

Replacing $\log_2 \omega(s,t)$ by $\log_2 \omega(0,1)$, we have proved (11) for level $\lfloor p \rfloor + 1$. The remaining can be proved by induction. Suppose (11) holds for $k = \lfloor p \rfloor + 1, \cdots, n$, then by breaking the sum into parts $1, \cdots \lfloor p \rfloor$ and $\lfloor p \rfloor + 1, \cdots n$, we have

$$\sum_{k=1}^{n} \left\| R_{u_{j-1},u_j}^k \otimes X_{u_j,u_{j+1}}^{n+1-k} \right\| < \frac{\epsilon p \cdot 2^{\frac{1-\{p\}}{p}}}{\beta^2(\frac{n+1}{p})!} \left(1 + \frac{p}{1-\{p\}} + \log_2 \frac{\omega(s,t)}{\epsilon^{(1-\{p\})/p}}\right) [\frac{1}{2^K}\omega(s,t)]^{\frac{n+\{p\}}{p}},$$

and similar bounds hold for $\sum_{k=1}^n \left\| X_{u_{j-1},u_j}^k \otimes R_{u_j,u_{j+1}}^{n+1-k} \right\|$ and $\sum_{k=1}^n \left\| R_{u_{j-1},u_j}^k \otimes R_{u_j,u_{j+1}}^{n+1-k} \right\|$. Thus, same as before, by successively dropping the 2^K points in $\mathcal{P}_{K+1} - \mathcal{P}_K$, we get

$$\begin{aligned}&\left\|(\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1}\right\| - \left\|(\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n+1}\right\| \\ &\leq \frac{4\epsilon p \cdot 2^{\frac{1-\{p\}}{p}}}{\beta^2(\frac{n+1}{p})!} \left(1 + \frac{p}{1-\{p\}} + \log_2 \frac{\omega(0,1)}{\epsilon^{(1-\{p\})/p}}\right) \omega(s,t)^{\frac{n+\{p\}}{p}} (\frac{1}{2^K})^{\frac{n-\lfloor p \rfloor}{p}}.\end{aligned}$$

Since now the exponent $\frac{n-\lfloor p \rfloor}{p} > 0$, by summing over K from 0 to $+\infty$, we conclude that

$$\left\| X_{s,t}^{n+1} - Y_{s,t}^{n+1} \right\| < \epsilon \left(1 + \frac{p}{1-\{p\}} + \log_2 \frac{\omega(0,1)}{\epsilon^{(1-\{p\})/p}} \right) \frac{\omega(s,t)^{\frac{n+\{p\}}{p}}}{\beta(\frac{n+1}{p})!},$$

as long as $\beta > 4p \cdot 2^{\frac{1-\{p\}}{p}} / [1 - (\frac{1}{2})^{\frac{1}{p}}]$, which clearly satisfies (10). Thus, we have completed the induction, and proved (11) for all $k \geq \lfloor p \rfloor$.

Case 3. $\delta \in (1 - \{p\}, 1]$.

This is possible only if p is non-integer. The proof is similar to the case $\delta = 1 - \{p\}$. In this case, applying Lemma 3.6 to $n + 1 = \lfloor p \rfloor + 1$, we have

$$\begin{aligned} \left\| (\hat{X}_{s,t}^{\mathcal{P}_N} - \hat{Y}_{s,t}^{\mathcal{P}_N})^{n+1} \right\| &\leq \frac{\epsilon p}{\beta^2 (\frac{n+1}{p})!} \cdot 2^{\frac{2p+\delta}{p}} \cdot \omega(s,t)^{\frac{n+1-\delta}{p}} \sum_{K=0}^{N-1} 2^{\frac{K}{p}(\delta + \{p\} - 1)} \\ &\leq \frac{\epsilon p}{\beta^2 (\frac{n+1}{p})!} \cdot [2^{\frac{2p+\delta}{p}} / (2^{\frac{\delta - 1 + \{p\}}{p}} - 1)] \cdot \omega(s,t)^{\frac{n+1-\delta}{p}} \cdot 2^{\frac{N}{p}(\delta - 1 + \{p\})}. \end{aligned}$$

By using the second inequality in (7), we have

$$\left\| (\hat{X}_{s,t}^{\mathcal{P}_N} - \hat{Y}_{s,t}^{\mathcal{P}_N})^{n+1} \right\| \leq \frac{p}{\beta^2 (\frac{n+1}{p})!} \cdot [2^{\frac{3p+\delta}{p}} / 1 - (\frac{1}{2})^{\frac{\delta - 1 + \{p\}}{p}}] \cdot \epsilon^{\frac{1-\{p\}}{\delta}} \omega(s,t). \tag{19}$$

On the other hand, similar to the previous case, we have

$$\sum_{K=N}^{+\infty} \left(\left\| (\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1} \right\| - \left\| (\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n+1} \right\| \right) \leq \frac{2p\omega(s,t)^{\frac{n+1}{p}}}{\beta^2 (\frac{n+1}{p})!} \sum_{K=N}^{+\infty} (\frac{1}{2^K})^{\frac{1-\{p\}}{p}}.$$

Using the first inequality in (7), we get

$$\sum_{K=N}^{+\infty} \left(\left\| (\hat{X}_{s,t}^{\mathcal{P}_{K+1}} - \hat{Y}_{s,t}^{\mathcal{P}_{K+1}})^{n+1} \right\| - \left\| (\hat{X}_{s,t}^{\mathcal{P}_K} - \hat{Y}_{s,t}^{\mathcal{P}_K})^{n+1} \right\| \right) \leq \frac{2p\epsilon^{\frac{1-\{p\}}{\delta}}}{1 - (\frac{1}{2})^{\frac{1-\{p\}}{p}}} \cdot \frac{\omega(s,t)}{\beta^2 (\frac{n+1}{p})!}. \tag{20}$$

Combining (12), (19) and (20) we get

$$\left\| X_{s,t}^{n+1} - Y_{s,t}^{n+1} \right\| < \epsilon^{\frac{1-\{p\}}{\delta}} \cdot \frac{\omega(s,t)^{\frac{n+\{p\}}{p}}}{\beta(\frac{n+1}{p})!},$$

where $n = \lfloor p \rfloor + 1$. For $k \geq \lfloor p \rfloor + 2$, we can apply the same induction procedure as in the previous case. Thus, we prove (13) for all $k \geq \lfloor p \rfloor + 1$.

□

5 An application

We now explain briefly how our estimates apply to the problem mentioned in the introduction (see equations (3) and (4) and assumption (5)). Since A is a linear map, the solutions can be written as

$$x_t = \sum_{n=0}^{+\infty} A^{*n} x_0 \int_{0<u_1<\cdots<u_n<t} d\gamma_{u_1} \cdots d\gamma_{u_t},$$

and

$$y_t = \sum_{n=0}^{+\infty} A^{*n} y_0 \int_{0<u_1<\cdots<u_n<t} d\tilde{\gamma}_{u_1} \cdots d\tilde{\gamma}_{u_t},$$

Let X and Y denote the signatures of γ and $\tilde{\gamma}$, then Theorem 1.1 implies that

$$\left\| X_{s,t}^n - Y_{s,t}^n \right\| < \epsilon(1 + \log_2 \frac{C}{\epsilon}) \cdot \frac{\omega(s,t)^{n-1}}{\beta n!}, \qquad \forall n \geq 2, \tag{21}$$

where $C \leq \omega(0,1)$ is a generic constant. Let $x_{s,t} = x_t - x_s$ and $y_{s,t} = y_t - y_s$, then we have

$$\begin{aligned} |x_{s,t} - y_{s,t}| &\leq \sum_{n=0}^{+\infty} \|A\|^n \left\| X_{s,t}^n - Y_{s,t}^n \right\| \\ &\leq \|A\| \cdot \left\| X_{s,t}^1 - Y_{s,t}^1 \right\| + \sum_{n=2}^{+\infty} \|A\|^n \epsilon(1 + \log_2 \frac{C}{\epsilon}) \cdot \frac{\omega(s,t)^{n-1}}{\beta(n-1)!} \\ &\leq 2\|A\| \min\{\epsilon, \omega(s,t)\} + \epsilon(1 + \log_2 \frac{C}{\epsilon}) \frac{|A\|}{\beta} (e^{|A\|\omega(s,t)} - 1). \end{aligned}$$

In particular, the difference of the two solutions $|x_t - y_t|$ are of order ϵ up to a logarithmic correction, uniformly in $t \in [0,1]$.

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